Causal structure of the entanglement renormalization ansatz

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We show that the multi-scale entanglement renormalization ansatz (MERA) can be reformulated in terms of a causality constraint on discrete quantum dynamics. This causal structure is that of de Sitter space with a flat spacelike boundary, where the volume of a spacetime region corresponds to the number of variational parameters it contains. This result clarifies the nature of the ansatz, and suggests a generalization to quantum field theory. It also constitutes an independent justification of the connection between MERA and hyperbolic geometry which was proposed as a concrete implementation of the AdS-CFT correspondence.

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The multi-scale entanglement renormalization ansatz (MERA), introduced in [1], has been successfully used to model the physics of many low-dimensional strongly correlated quantum many-body systems [1–9], thanks to its ability to simultaneously represent correlations at widely different length scales. It is defined as the set of states which can be output by a quantum circuit with fixed input and a given gate structure. The components of the gates are the variational parameters.

In this paper, we show that only one general property of the circuit is necessary to define the ansatz: its causal structure, which is defined as a constraint on the flow of information. By clarifying how and when the specific details of the circuit are unimportant, this approach unifies and simplifies the formulations of a MERA, and opens the way to a deeper theoretical study of the ansatz, such as its relation to other theories of renormalization [10]. We show that the causal structure of MERA is essentially identical to that of de Sitter spacetime, i.e. an expanding universe undergoing an exponential multiplication of its local degrees of freedom, such as the inflationary period in cosmology.

This naturally points to quantum field theory on de Sitter space as a continuous generalization of MERA able to represent the state of critical quantum field theories. We show that this proposal is compatible with that of Ref. [11], but better constrained, and amenable to the tools developed in the context of quantum cosmology. Additionally, this result constitutes a connection between MERA and hyperbolic geometry (the Euclidean form of both anti de Sitter and de Sitter spacetimes), which is independent from the arguments previously proposed in connection with the AdS-CFT correspondence [12, 13].

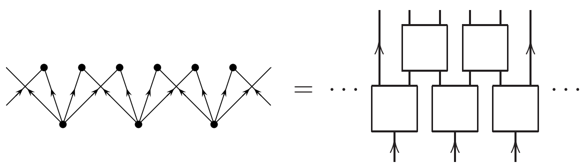
MERA FROM CAUSAL ORDER

One step of a circuit defining a MERA takes a quantum state defined on a coarse-grained lattice and isometrically maps it into the larger Hilbert space of a finer lattice. These isometric steps are also required to act locally. This implies a finite speed of information propagation in the circuit. Combined with the exponential nature of the successive coarse-graining operations, this means that the expectation value of a local observable can be evaluated in a time logarithmic in the lattice size. In order to see this, note that an expectation value can be evaluated by evolving the observable "back in time" in the Heisenberg picture and then computing the expectation value between the initial fiducial state and the resulting observable. When talking about locality it is enlightening to adopt the Heisenberg picture because there exists an unambiguous concept of a local observable: one which acts nontrivially only on certain lattice sites. The locality of the isometries then implies that an observable local to a region Σ of the lattice is pulled back to an observable which is itself local with respect to a region Σ' that is in the *causal past* of Σ [2]. This map is the dual of a local quantum channel mapping states defined on Σ' to states on Σ. In performing this operation, the rest of the isometry can be completely ignored. Furthermore, the coarse-graining is such that the Hilbert space dimension associated with the causal past Σ' is no larger than that of Σ, and hence the computational load can only decrease at each step, and is independent of the lattice size.

This suggests that, in defining the ansatz, we could replace the ad-hoc requirement that each isometric step have a particular gate structure, and just require it to pull back local observables (on Σ) to local observables (on Σ'). This property is precisely one of *causality* as it is equivalent to stating that the degrees of freedom outside Σ' cannot influence those inside Σ through one step of the dynamics [14]. But is such a property sufficient to obtain an efficient local parameterization of the isometries? It was shown by Arrighi *et al.* [15] in the context of a unitary dynamical step U that such causal constraints are sufficient and necessary for U to be implementable as a circuit of local operations, with some commutativity constraints between them. For a MERA, however, we need each step to be isometric, for which the causality constraints are not sufficient to render the parameterization efficient. Indeed, the isometry can always acausally produce a state correlated over arbitrary distances. Here we introduce a stronger causality requirement which applies to isometric maps as well as to generic quantum

channels. We demand that the map can be implemented by creating a separable state on extra systems at each lattice sites (with no constraint on their dimensions), followed by a causal unitary map. We will say that such a map is *purely* causal rather than just causal. The result of Ref. [15] then trivially extends to this stronger requirement, showing that a purely causal channel can be implemented with local operations.

We will represent a given causal relation between two lattices as a bipartite graph, where the presence of an edge between a site of the input lattice and a site of the output lattice indicates a possible causal influence. The causality constraints therefore are embodied in those pairs of vertices which are *not* connected by an edge. We also interpret a particular causality relation as representing the set of isometries which are purely causal with respect to it. We will show that



(1)

where time flows upward, and the right-hand side represent the set of isometries which can be implemented by the circuit (or tensor network) obtained by replacing each box by an isometry. This is precisely one step of the binary MERA [1, 2], except for the fact that in our case there is no constraint on the dimension of the Hilbert spaces associated with the intermediate wires. However, the fact that each box must be an isometry limits their input of dimension to that of their output.

Below we also show that the ternary MERA [2] is equivalent to such a natural causality constraint. More generally, our prescription together with the constructive localizability result introduced in Ref. [15] allows for the construction of circuits with equivalent properties on arbitrary lattices, including lattices embedded in higher dimensional spaces. We defer the explanation of how a general circuit equivalent to a given causality structure can be designed to Ref. [15].

Our approach works just as well if we allow each step to be implemented by a quantum channel rather than just an isometry, hence allowing in principle for the characterization of mixed states with long range correlations, such as critical thermal states. If the state to be described is classical one may furthermore constrain the local quantum channels to be stochastic maps (i.e. to map diagonal matrices to diagonal matrices).

CONNECTION WITH DE SITTER SPACE

Let us call an “event” a lattice site at a given coarse-graining step. Each event is associated with the Hilbert space dimension of the corresponding lattice site. The causal relations between successive coarse-graining generates a partial order between any two events, i.e., $A < B$ if A is in the causal past of B. Furthermore the original causal relation can be recovered uniquely from the partial order by noting that it is defined by the causal *links*: pairs of related events with no events “in between”, i.e., (A, B) forms a link if $A < B$ and there is no C such that $A < C < B$.

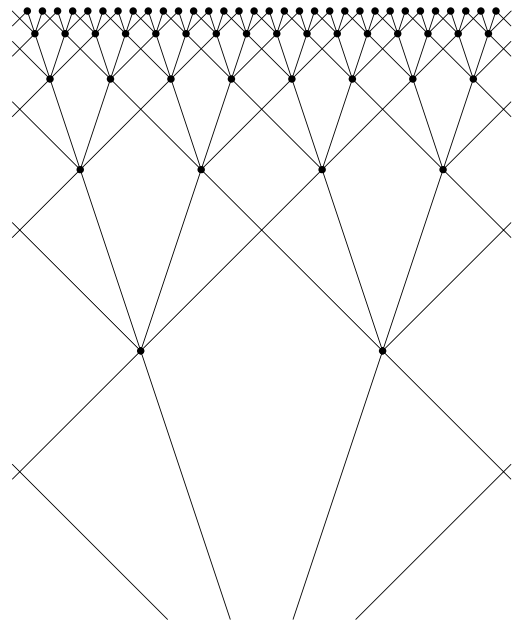


FIG. 1: Partial ordered set corresponding to the binary MERA in one dimension, embedded in $\mathbb{R}^2$ such that the speed of light is equal to 1 everywhere. The output lattice is the top row of dots and time flows upward. The black circles are events and the lines segments are causal links.

Therefore, one can recover the MERA simply from the causal order between the set of events, together with the dimensions of their assigned Hilbert spaces. For instance, the usual binary MERA [1] is implied by the partial order shown in Figure 1. Such “causal sets” have been studied before as discrete models of spacetime [16, 17]. The idea follows from the fact that the geometry of a manifold with Lorentzian signature can be recovered exactly from the partial order between events induced by the metric, together with the volume form. The causal order directly makes sense for a discrete spacetime. For the volume form, a natural postulate is that it corresponds to the counting of events. In our case, assuming for simplicity that all events are associated with the same Hilbert space dimension, the number of events in a given spacetime region is proportional to the number of variational parameters, thank to the local representability result.

In order to see what metric a MERA on a d-dimensional lattice may correspond to, the easiest is to

first parameterize its events by coordinates in which the speed of light is constant (and equal to 1), i.e. in a spacetime with metric $ds^2 = f(t, x_1, \dots, x_d)\big(-dt^2 + \sum_i dx_i^2\big)$. We suppose that each coarse-graining increases the lattice spacing by a factor a, and that sites at the $(k+1)$th coarse-graining step have a causal influence on the sites of the kth step within a radius ra^k. Then a constant speed of light (equal to 1) is achieved by embedding the kth coarse-graining at time $t = -ra^k/(a-1)$. We choose it negative so that it increases chronologically with the quantum computation, outputting the final state at time $t_0 = -r/(a-1)$. In order to determine the conformal factor $f(t, x_1, \dots, x_d)$, we postulate that in coordinates where our lattices are equally spaced in time, and renormalized, the volume form should be constant. This makes precise the idea that the number of events in a given region of spacetime should be proportional to the volume of that region. Such coordinates must be of the form $\tau = -\alpha \log[t/t_0]$ and $\zeta_i = -\beta x_i/t$. The constraint is then satisfied by picking $f(t) = (\alpha/t)^2$. Also, choosing $t_0 = -r/(a-1)$ puts the output boundary $k = 0$ at $\tau = 0$, and $\beta = \alpha$ normalizes the volume element. In the coordinates $(\tau, \zeta_1, \dots, \zeta_d)$ the metric is then

$$ds^2 = \left(\frac{\rho^2}{\alpha^2} - 1\right) d\tau^2 - 2\frac{\rho}{\alpha}\, d\rho\, d\tau + \sum_i d\zeta_i^2.$$

where $\rho^2 = \sum_i \zeta_i^2$, and the volume form has component $\sqrt{|\det g|} = 1$. In the conformally flat coordinates this is

$$ds^2 = \left(\frac{\alpha}{t}\right)^2 \big(-dt^2 + \sum_i dx_i^2\big).$$

This metric is that of de Sitter space. Another common coordinate system is given by the time coordinate τ together with $\xi_i = -\alpha x_i/t_0$, so that

$$ds^2 = -d\tau^2 + e^{2\tau/\alpha} \sum_i d\xi_i^2.$$

Basic properties of the MERA can be deduced from considering past lightcones in the coordinates $(\tau, \zeta_1, \dots, \zeta_d)$ with constant volume form. The lightlike worldlines can be deduced by applying the coordinate change to the Minkowski ones. They are all of the form

$$\zeta_i(\tau) = -\alpha u_i (1 - e^{\tau/\alpha}) + \zeta_i(0) e^{\tau/\alpha}$$

where u_i a unit vector. We see that the causal past of any bounded region converges in the infinite past $\tau \to -\infty$ to the ball of radius α (the cosmological horizon), which contains a fixed number of lattice sites at any given time. Figure 2 illustrates this phenomenon. It shows the causal past associated with the computation of the correlation function between two local observables for $d = 1$. The log-dimension of the tensors that one needs to contract at each time step is proportional to the spacelike volume of the shaded area at that time.

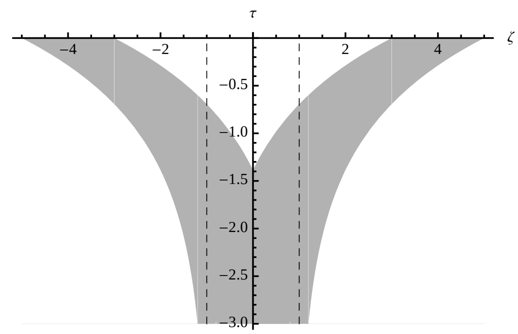


FIG. 2: The shaded area is the causal past of two disconnected regions of the $\tau = 0$ spacelike surface in the static coordinates (τ, ζ). The dashed lines indicate the horizon at $|\zeta| = \alpha = 1$.

This analysis yields a possible approach to building a continuous MERA: by parameterizing quantum field theories on de Sitter space. In order to compare this to the "cMERA" proposed by Haegeman *et al.* [11], we consider a quantum scalar field, with Lagrangian density

$$\mathcal{L} = \frac{1}{2}\sqrt{|\det g|}[-g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - V(\phi)].$$

Given our prescriptions, the coordinate system using in Ref. [11] must correspond to our static coordinate system $(\tau, \zeta_1, \dots)$ given that the same cutoff is used at each time step. This Lagrangian can be quantized by standard methods. If $\hat{\pi}(\zeta_1, \dots, \zeta_d)$ are the canonical conjugates to the field operators $\hat{\phi}(\zeta_1, \dots, \zeta_d)$, then we obtain the Hamiltonian $H = K + L$, where

$$K = \frac{1}{2}\int d^d\zeta \Big[\hat{\pi}^2 + \sum_i (\hat{\phi}_{,i})^2 + V(\hat{\phi})\Big]$$

is the Hamiltonian for the field on flat spacetime and

$$L = -\frac{1}{\alpha}\int d^d\zeta \sum_i \frac{1}{2}\Big[\hat{\pi}\zeta_i\hat{\phi}_{,i} + \zeta_i\hat{\phi}_{,i}\hat{\pi}\Big].$$

generates the expansion of space. One can easily check that, with canonical commutation relations between $\hat{\pi}$ and $\hat{\phi}$, this yields the right Heisenberg equations of motion. This structure for H is compatible with that proposed by Haegeman *et al.*, although our approach yields stronger constraints given that it requires exact causality.

We note also that the Euclidean form of de Sitter space is identical to that of anti de Sitter space, namely hyperbolic space. There has been great interest recently in MERA as providing a possibly concrete form of the AdS-CFT correspondence for its ability to represent critical scale-invariant states and the similarity in which entanglement entropy is calculated using a minimal surface in the bulk [12, 13]. This suggests an interpretation where the MERA circuit represents a field theory on AdS spacetime, whereas the state it describes is a

CFT on its boundary. The Euclidean form of our field theory lives precisely on the same spacetime, and with the same boundary, as the Euclidean form of AdS spacetime. Therefore our approach provides an independent way of relating MERA to hyperbolic geometry.

CAUSALITY AND LOCALITY

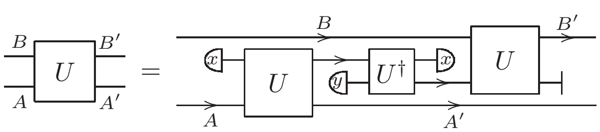
We will now sketch the proof of the statement represented by Equ. 1. Namely, that the set of isometries which are purely causal with respect to the causality relation represented on the left-hand side is equal to the set of isometries which can be formed by replacing each box of the right hand side by an isometry (without constraint on the dimension of the Hilbert spaces associated with the middle wires). First we consider a unitary map U whose inputs are grouped into systems A and B, and outputs are grouped into systems A' and B',with the constraint that B cannot influence A'. This means that in the Heisenberg picture, any operator X acting on system A' is mapped to an operator Y acting only on system A, i.e. $U^\dagger(X_{A'} \otimes \mathbf{1}_{B'})U = Y_A \otimes \mathbf{1}_B$, which can be rewritten as

$$(X_{A'} \otimes \mathbf{1}_{B'})\, U = U\,(Y_A \otimes \mathbf{1}_B). \tag{2}$$

This implies that for any pure state $|x\rangle$ of B,

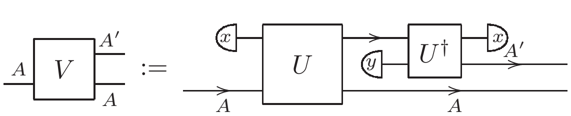
$$Y = (\mathbf{1} \otimes \langle x|)\, U^\dagger\, (X \otimes \mathbf{1})\, U\, (\mathbf{1} \otimes |x\rangle).$$

The trick, inspired by Ref. [15], is to replace the local operator X by a swap between A' and a new system C in Equation (2). If we initialize the system C to an arbitrary state $|y\rangle$ and trace it out after the action of U and the swap, the left hand side becomes simply U. This yields the expansion

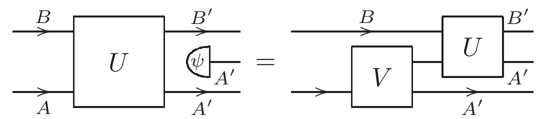


(3)

where the states $|x\rangle$ and $|y\rangle$ can be chosen arbitrarily. This is the only algebraic property that we will need. The vertical bar ending the fourth wire means that this system is traced out: hence both sides of this equation represent channels rather than just operators. The channel on the left-hand side is just $\rho \mapsto U\rho U^\dagger$: being unitary, it is a minimal Stinespring dilation of the channel on the right hand side. From the unicity of the Stinespring dilation of a channel, the right-hand side has also only one Kraus operator. To find its precise form, first note that the operator

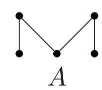


is an isometry as can be checked by tracing out A' and B' on both sides of Equation (3). Then the Stinespring dilation theorem tells us that there is an isometry (here just a ket) $|\psi\rangle$ embedding $\mathbb{C}$ into the Hilbert space of the system A' such that

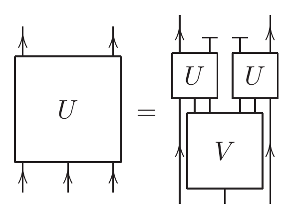


It follows that we can replace the channel $\mathcal{N}(\rho) := \mathrm{tr}_{A'} U\rho U^\dagger$ in Equ. (3) by $\rho \mapsto X\rho X^\dagger$, with $X := (\mathbf{1}_{B'} \otimes \langle\psi|_{A'})U$. Furthermore, since the whole expression must be unitary, and hence trace-preserving, the operator X is isometric when restricted to its possible inputs in the circuit, and can therefore be replaced by an isometry.

This can be used to parameterize the classes of unitary maps causal with respect to a relation like that of Equ. (1) as follows: we start by grouping all the inputs (resp. outputs) which have the set of children (resp. parents) to obtain a new causal relation on the grouped systems. If the resulting graph is such that removing one particular input A breaks it into two independent parts, then the remaining inputs and outputs can be grouped so as to satisfy the causality relation



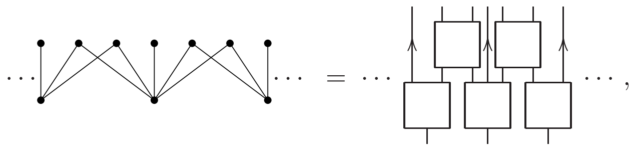
This represents two causality constraints (i.e. missing links). By applying the instance of Equ. 3 allowed by one of the constraint, and then again on the first instance of U in the circuit for the other constraint, we obtain that



for some isometry V. This scheme can be applied recursively on the remaining copies of U, until the circuit respects all the causality constraints.

If we lift the restriction that our computational step be unitary, and assume instead that it is an isometry (as is required for a MERA) or more generally a quantum channel, then we demand that it can be represented by a unitary interaction with a local environment, such that the unitary map respects the same causal relation. We also require that the environment's initial state is separable. We can then apply our procedure to this unitary map to show that it has a local representation. In this way, one obtains the result express in Equ. (1). This

method also works for the ternary MERA, showing that



with the same disclaimer about the fact that the dimensionality of intermediate wires are not constrained, but limited by the fact that the boxes must represent isometries.

As mentioned in the introduction, for more general causality relations, in particular as applied to higher-dimensional lattices, one must use the general prescriptions introduced in Ref. [15]. For completeness, we describe their construction here. A unitary map respecting a general set of causality constraints can be implemented locally as follows: one first produce ancillas matching the number and dimensions of all the *output* systems, initialized with an arbitrary product state. We may then apply, for each possible output system, a unitary interaction between the ancilla corresponding to that output and its parent *inputs*, i.e. the input from which it is allowed to receive information from according to the causality constraints. The only condition on these local unitary interactions is that they all commute with each other. One must then trace out all the input systems, and be left with the ancilla, which we identify as the output systems. The global unitary respects the causality constraints if and only if it can be implemented in this manner.

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